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The Calogero–Moser equation system and the ensemble average in the Gaussian ensembles

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Abstract

From random matrix theory it is known that for special values of the coupling constant the Calogero–Moser (CM) equation system is nothing but the radial part of a generalized harmonic oscillator Schrödinger equation. This allows an immediate construction of the solutions by means of a Rodriguez relation. The results are easily generalized to arbitrary values of the coupling constant. By this the CM equations become nearly trivial. As an application an expansion for $\langle e^{\text{Tr}(XY)} \rangle$ in terms of eigenfunctions of the CM equation system is obtained, where X and Y are matrices taken from one of the Gaussian ensembles, and the brackets denote an average over the angular variables.

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1. Introduction

There are only a very limited number of many-body systems allowing explicit solutions [1]. This is why the Calogero–Moser (CM) equation system from the very beginning has attracted much interest. In its original version the system Hamiltonian reads

$$\hat{H} = -\frac{1}{2} \sum_{n=1}^N \frac{\partial^2}{\partial \hat{x}_n^2} + \sum_{n>m} \left[\frac{1}{2} (\hat{x}_n - \hat{x}_m)^2 + \frac{g}{(\hat{x}_n - \hat{x}_m)^2} \right]. \quad (1)$$

The parameter g describes the strength of the pair interaction potential decreasing with the square of the distance. The eigenfunctions of \hat{H} , given already by Calogero in his original work [2], may be written as

$$\psi_{nk}(\hat{x}) = z^{\beta/2} \varphi_{nk}(r) P_k(\hat{x}) \quad (2)$$

where $\beta = 1 + \sqrt{1 + 4g}$, and

$$z = \prod_{n>m} |\hat{x}_n - \hat{x}_m| \quad r^2 = \sum_{n>m} (\hat{x}_n - \hat{x}_m)^2. \quad (3)$$

\hat{x} is a short-hand notation for $(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_N)$. The radial eigenfunctions $\varphi_{nk}(r)$ read

$$\varphi_{nk}(r) = L_n^{(b)}\left(\frac{1}{2}r^2\right)e^{-\frac{1}{2}r^2} \quad (4)$$

where $L_n^{(b)}(x)$ is a generalized Laguerre polynomial ($n = 0, 1, \dots$) and

$$b = k + \frac{1}{2}(N - 2) + \frac{\beta}{4}N(N - 1) \quad (5)$$

(there is a misprint in equation (2.13) of [2] for b). $P_k(\hat{x})$ is a polynomial of degree k , totally symmetric in the variables \hat{x}_n . It is obtained as a solution of

$$\left[\sum_n \frac{\partial^2}{\partial \hat{x}_n^2} + \beta \sum_{n \neq m} \frac{1}{\hat{x}_n - \hat{x}_m} \frac{\partial}{\partial \hat{x}_n} \right] P_k(\hat{x}) = 0. \quad (6)$$

The eigenvalues corresponding to $\psi_{nk}(\hat{x})$ are given by

$$E_{nk} = \frac{1}{2}(N - 1) + \frac{\beta}{4}N(N - 1) + 2n + k. \quad (7)$$

Apart from a shift of the energy zero and differing multiplicities of the energy levels this is just the spectrum of an harmonic oscillator.

An equivalent form for the Hamiltonian, which is more convenient in the present context, is given by

$$H = \frac{1}{2} \sum_{n=1}^N \left(-\frac{\partial^2}{\partial x_n^2} + x_n^2 \right) - \frac{\beta}{2} \sum_{n \neq m} \frac{1}{x_n - x_m} \frac{\partial}{\partial x_n}. \quad (8)$$

\hat{H} is obtained from H by introducing centre of mass variables $\hat{x}_n = x_n - \frac{1}{N} \sum_n x_n$ and applying the transformation

$$\hat{H} = z^{-\beta/2} H z^{\beta/2} - H_{c.m.} \quad (9)$$

where $H_{c.m.}$ is the centre of mass Hamiltonian.

From the very beginning there was the strong suspicion that there must be a close connection between the CM system and the quantum mechanics of the harmonic oscillator, just from the inspection of the spectrum (7). But it lasted for 30 years, until the complete equivalence of the two systems was proved by Gurappa and Panigrahi [3]. For the proof a quite evolved operator technique was applied. There were many futile attempts as well to find expressions for the polynomial $P_k(\hat{x})$, until recently an explicit construction of the solutions of the CM system has been achieved by Ujino and Wadati [4].

In this paper it will be demonstrated that there is a much simpler connection between the quantum mechanics of the harmonic oscillator and the CM system. In fact for the special cases $\beta = 1, 2, 4$ this connection will show up to be completely trivial.

The key ingredient to this result comes from random matrix theory, where the CM equations enter via the calculation of averages in the Gaussian ensembles (see e.g. [5, 6]). Since this background probably is not known to all of the experts engaged in the CM equations, a short recapitulation of the aspects relevant in the present context is given in section 2. In section 3 the equivalence of the CM system with the Schrödinger equation of the harmonic oscillator is established for the cases $\beta = 1, 2, 4$. This result is generalized to arbitrary values of β in section 4. In section 5 an application to the calculation of averages in the Gaussian ensembles is presented. The paper ends with a short discussion in section 6.

2. The averaging problem in the Gaussian ensembles

An issue of central importance in random matrix theory is the calculation of averages over products and ratios of spectral determinants,

$$M_\nu(E_i) = \left\langle \prod_i |E_i - H|^{v_i} \right\rangle \quad (10)$$

where the matrix elements of H are assumed to be Gaussian distributed, and the v_i may take positive and negative integer values. The standard way to calculate such averages uses supersymmetry techniques. In this approach after some steps an integral of the type

$$I(Y) = \int d[x] e^{i\text{Tr}(XY) - \text{Tr}(F(X))} \quad (11)$$

is met, where X, Y are $N \times N$ supermatrices. The integral is both over the symmetric and the antisymmetric components of X (see [7] for more details). Diagonalizing X ,

$$X = R X_D R^{-1} \quad (12)$$

where $X_D = \text{diag}(x_1, \dots, x_N)$, one has

$$I(Y) = \int \left(\prod_i dx_i e^{-F(x_i)} \right) B(x) f(x, y). \quad (13)$$

$B(x)$ is the radial part of the functional determinant, and $f(x, y)$ is given by

$$f(x, y) = \langle e^{i\text{Tr}(XY)} \rangle \quad (14)$$

where the brackets denote the average over the angular variables. From the invariance property of the trace it is obvious that $f(x, y)$ depends only on the eigenvalues of X and Y , but not on the respective angular variables.

The practical applicability of supersymmetry techniques relies on convenient expressions for $f(x, y)$. For Hamiltonians taken from the Gaussian unitary ensemble (GUE), the angular average has been calculated by Itzykson and Zuber [8] for ordinary matrices. Their result has been generalized to supermatrices by Guhr [9]. Explicit expressions for the average (10) over spectral determinants have been given by Fyodorov and Strahov [10].

For the Gaussian orthogonal ensemble (GOE), much more important from the practical point of view, and the Gaussian symplectic ensemble (GSE) the situation is unsatisfactory. Expressions for the ensemble average of the negative moments of the spectral determinant have been given by Fyodorov and Keating [11]. Brezin and Hikami were able to express averages over products of spectral determinants in terms of complicated integrals over determinants of quaternionic matrices [12]. Recursive relations for $f(x, y)$ with respect to the rank N have been given by Guhr and Köhler [5, 13].

For all ensembles there are expansions of the type

$$f(x, y) = \sum_n a_n^{2/\beta} P_n^{2/\beta}(x) P_n^{2/\beta}(y) \quad (15)$$

where the $P_n^{2/\beta}(x)$ are Jack polynomials [14, 15]. The sum is over all partitions $n = (n_1, \dots, n_N)$, where $|n| = \sum_i n_i = n$ is the degree of the polynomial. The polynomials are totally symmetric in the variables, and are orthogonal on the interval $[-\frac{1}{2}, \frac{1}{2}]$ with the weight function $\prod_{n>m} |x_n - x_m|^\beta$,

$$\int_{-1/2}^{1/2} dx_1 \cdots \int_{-1/2}^{1/2} dx_n \prod_{n>m} |x_n - x_m|^\beta P_n^{2/\beta}(x) P_m^{2/\beta}(x) \sim \delta_{nm} \quad (16)$$

(see e.g. [16]).

Unfortunately, the results given in the mentioned references are not very convenient for the practical application. This is why for the GOE, in contrast to the GUE, only a quite small number of analytic results on ensemble averages are available. On the other hand, there are a number of experiments, in particular in chaotic microwave cavities performed in the author's group, where exactly such averages are needed [17, 18]. This was the original motivation for this work.

3. The Calogero–Moser equations for $\beta = 1, 2, 4$

$f(x, y)$, considered as a function of x , obeys the differential equation

$$(\Delta + \text{Tr } Y^2)f(x, y) = 0 \quad (17)$$

where

$$\text{Tr } Y^2 = \sum_{ij} Y_{ij} Y_{ji} = \sum_n y_n^2 \quad (18)$$

and

$$\Delta = \sum_{ij} \frac{\partial^2}{\partial x_{ij} \partial x_{ji}} \quad (19)$$

is a generalized Laplace operator. Since $f(x, y)$ is independent of the angular variables of X , we may restrict the Laplace operator to its radial part. It is well known from numerous papers on random matrix theory (see e.g. the references cited above), how to separate the Laplace operator into its radial and angular parts, but for the convenience of readers not familiar with the topic the essential steps shall be repeated.

To keep the discussion simple we assume that all operators are Hermitian, i.e. both summation indices i, j in equations (18) and (19) run from 1 to N . The generalization to the other two symmetry classes is straightforward though some care has to be taken to avoid double counting. For symmetric operators, e.g. in all traces the summations have to be restricted to $i \leq j$. In addition we restrict ourselves to ordinary commuting variables. The extension to supersymmetric variables, just as it was demonstrated in [9] for the GUE, is straightforward as well.

We start with the somewhat more general equation

$$\frac{1}{2}(-\Delta + \text{Tr } X^2)\psi_n(X) = E_n \psi_n \quad (20)$$

which is nothing but the Schrödinger equation for an independent superposition of harmonic oscillators. Its solution may thus be expressed in terms of products of single harmonic oscillator eigenfunctions, or, using Rodriguez' formula for the Hermite polynomials, as

$$\begin{aligned} \psi_{n_{11} \dots n_{NN}}(X) &\sim e^{\frac{1}{2} \text{Tr } X^2} \left[\prod_{ij} (-\nabla_{ij})^{n_{ij}} \right] e^{-\text{Tr } X^2} \\ &\sim \left[\prod_{ij} (x_{ij} - \nabla_{ij})^{n_{ij}} \right] e^{-\frac{1}{2} \text{Tr } X^2}. \end{aligned} \quad (21)$$

The corresponding eigenenergies are

$$E_n = \sum_{ij} \left(n_{ij} + \frac{1}{2} \right) = \sum_{ij} n_{ij} + \frac{N^2}{2}. \quad (22)$$

The subset of solutions, depending on the radial variables only, is obtained from equation (21) as

$$\psi_n(x) \sim \left(\prod_i A_+^{n_i} \right) e^{-\frac{1}{2}\text{Tr}X^2} \quad (23)$$

where n is an abbreviation for (n_1, n_2, \dots) , and

$$A_+^k = \text{Tr}[(X - \nabla)^k]_{\text{rad}} \quad (24)$$

is a generalized creation operator. The eigenenergies corresponding to $\psi_n(x)$ are given by

$$E_n = \sum_i n_i + \frac{N^2}{2} = |n| + \frac{N^2}{2} \quad (25)$$

where $|n| = \sum_i n_i$. Each partition (n_1, n_2, \dots) with the same $|n|$ thus yields an independent eigenfunction to the same eigenvalue.

Only by using invariance properties of the trace were we able to construct the radially symmetric solutions (23) of the generalized harmonic oscillator Schrödinger equation (20). These functions at the same time must be solutions of the radial part of the Schrödinger equation. Thus the radial part of the generalized Laplace operator entering equation (20) has to be determined. For the eigenfunctions in addition we need expressions for the radial part of $\text{Tr}(X - \nabla)^k$.

Details of the calculation can be found in appendix A. For the radial representation of the generalized creation operator we obtain

$$A_+^k = \sum_{nm} [(X_D - D)^k]_{nm} \quad (26)$$

where

$$D_{nm} = \delta_{nm} \left(\frac{\partial}{\partial x_n} + \frac{\beta}{2} \sum_k' \frac{1}{x_n - x_k} \right) - \frac{\beta}{2} \frac{1 - \delta_{nm}}{x_n - x_m}. \quad (27)$$

Equation (27) holds for all Gaussian ensembles with $\beta = 1, 2, 4$ for the GOE, the GUE and the GSE, respectively. For the radial part of the Laplace operator we get

$$\Delta_{\text{rad}} = \sum_n \frac{\partial^2}{\partial x_n^2} + \beta \sum_{n \neq m} \frac{1}{x_n - x_m} \frac{\partial}{\partial x_n}. \quad (28)$$

Note that this is exactly the operator entering equation (6) for the polynomials $P_k(x)$. For the radial part of the generalized oscillator Schrödinger equation (20) we have

$$H\psi_n(x) = E_n\psi_n(x) \quad (29)$$

where

$$H = \frac{1}{2} \sum_{n=1}^N \left(-\frac{\partial^2}{\partial x_n^2} + x_n^2 \right) - \frac{\beta}{2} \sum_{n \neq m} \frac{1}{x_n - x_m} \frac{\partial}{\partial x_n}. \quad (30)$$

This is identical with the CM Hamiltonian (8).

The derivation of this section has shown that for the special cases $\beta = 1, 2, 4$ the CM equation system is completely trivial. It is nothing but the radial part of the generalized harmonic oscillator Schrödinger equation (20). As a consequence the solutions are also obtained in a trivial way just by picking out all linear combinations of products of Hermitian polynomials which are totally symmetric in the variables.

4. The general case

For $\beta = 1, 2, 4$ the problem is thus completely solved. But all results can immediately be transferred to arbitrary values of β .

First we note that

$$\psi_0(x) \sim \exp\left(-\frac{1}{2} \sum_n x_n^2\right) \quad (31)$$

is the ground-state eigenfunction of the Calogero–Moser equation (29) for all values of β , with a ground-state energy given by

$$E_0 = \frac{N}{2} + \beta \frac{N(N-1)}{4}. \quad (32)$$

Second, the generalized creation operators obey the commutation rule

$$HA_+^k = A_+^k(H+k) \quad (33)$$

which is the generalization of a well-known relation for the one-dimensional harmonic oscillator (see appendix B). For $\beta = 1, 2, 4$ this follows trivially from equation (24). But the rule holds for arbitrary values of β , if only A_+^k is calculated from expression (26), which is well-defined for all values of β , and not from equation (24), which loses its meaning in the general case.

Since nothing but this commutation rule is needed for the generalized Rodriguez relations to hold, equation (23) can still be used to generate the eigenfunctions. The corresponding spectrum is thus not altered by varying β , apart from a shift of the ground-state energy, a fact noted already by Calogero [2].

5. A series expansion for $\langle e^{i\text{Tr}(XY)} \rangle$

Generalizing an expansion of e^{ixy} in terms of ordinary oscillator eigenfunctions we are now going to expand $f(x, y)$ in terms of the totally symmetric solutions (23) of the generalized Schrödinger equation (20)

$$f(x, y) = \langle e^{i\text{Tr}(XY)} \rangle = \sum_{nm} c_{nm} \psi_n(x) \psi_m(y). \quad (34)$$

In a sequence of self-explaining steps, including one integration by parts, we have

$$\begin{aligned} \int d[Y] \psi_n(y) f(x, y) &= \int d[Y] \psi_n(y) \langle e^{i\text{Tr}(XY)} \rangle \\ &= c_n \int d[Y] \left[\prod_i \text{Tr}(Y - \nabla_Y)^{n_i} e^{-\frac{1}{2}\text{Tr}Y^2} \right] e^{i\text{Tr}(XY)} \\ &= c_n \int d[Y] e^{-\frac{1}{2}\text{Tr}Y^2} \left[\prod_i \text{Tr}(Y + \nabla_Y)^{n_i} e^{i\text{Tr}(XY)} \right] \\ &= c_n i^{|n|} \left[\prod_i \text{Tr}(X - \nabla_X)^{n_i} \right] \int d[Y] e^{-\frac{1}{2}\text{Tr}Y^2} e^{i\text{Tr}(XY)} \\ &= c_n i^{|n|} \left[\prod_i \text{Tr}(X - \nabla_X)^{n_i} \right] e^{-\frac{1}{2}\text{Tr}X^2} \\ &= (2\pi)^{\frac{N^2}{2}} i^{|n|} \psi_n(x) \end{aligned} \quad (35)$$

where c_n is the normalization factor of the $\psi_n(x)$. The differential is given, either in Cartesian or radial-angular coordinates, by

$$d[Y] = \prod_{ij} dY_{ij} = \prod_{i < j} |y_i - y_j|^\beta \prod_i dy_i d\Omega_\beta \quad (36)$$

where $d\Omega_\beta$ is the angular part of the differential. Comparison with equation (34) yields

$$c_{nm} = (2\pi)^{\frac{N^2}{2}} i^{|n|} (\rho^{-1})_{nm} \quad (37)$$

where ρ is the matrix with the elements

$$\rho_{nm} = \int d[X] \psi_n(x) \psi_m(x). \quad (38)$$

Eigenfunctions belonging to different $|n|$ are orthogonal, but for degenerate eigenfunctions this is not necessarily the case. The matrix of expansion coefficients c_{nm} is thus block-diagonal, where each block corresponds to a given value of $|n|$.

The problem of diagonalization of the eigenfunctions belonging to the same $|n|$ has been solved by Ujino and Wadati in a series of papers [19–21]. The authors introduced a new type of totally symmetric polynomials $j_n^{2/\beta}(x)$ they called hidden Jack polynomials (or in short, Hi–Jack polynomials), which are obtained from linear combinations of the $\psi_n(x)$ introduced in equation (23). The Hi–Jack polynomials obey the orthogonality relation

$$\int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_N \prod_{k>l} |x_k - x_l|^\beta \exp\left(-\sum_k x_k^2\right) j_n^{2/\beta}(x) j_m^{2/\beta}(x) \sim \delta_{nm}. \quad (39)$$

Collecting the results we obtain an expansion of $f(x, y)$ in terms of Hi–Jack polynomials,

$$f(x, y) = \langle e^{i\text{Tr}(XY)} \rangle = (2\pi)^{\frac{N^2}{2}} \sum_n i^{|n|} \psi_n^{2/\beta}(x) \psi_m^{2/\beta}(y) \quad (40)$$

where the sum is over all partitions n , and

$$\psi_n^{2/\beta}(x) = c_n^{2/\beta} j_n^{2/\beta}(x) \exp\left(-\frac{1}{2} \sum_k x_k^2\right) \quad (41)$$

$c_n^{2/\beta}$ is a constant normalizing the integral (39) to 1 for $n = m$.

Equation (40) is an alternative to expansion (15) in terms of ordinary Jack polynomials. It has still to be checked whether this will lead to a real progress in the calculation of ensemble averages in the GOE. The formulae given in [19] for the Hi–Jack polynomials are quite complicated, but it does not seem hopeless that a more direct generation of the polynomials is possible, if the techniques applied in the present work are used.

6. Discussion

Using the fact that for special values of the coupling constant the CM equation system is nothing but the radial part of a generalized harmonic oscillator Schrödinger equation, the solution could be immediately constructed in terms of totally symmetric linear combinations of products of Hermitian polynomials. The generalization to arbitrary values of the coupling constant was then straightforward.

None of these results is really new. The equivalence of the CM equation system with the Schrödinger equation of the harmonic oscillator was suspected from the very beginning [2] and was proved recently [3]. Explicit solutions of the equation system, too, have been known for some time [4]. In fact solutions (23) derived in the present work are completely

equivalent to those given in [4]. The fact that the radial part of the generalized harmonic oscillator Schrödinger equation reduces to a special case of the CM equations, too, has been known from random matrix for several years.

But the experts working on the CM equations and on random matrix theory did not know much of each other as it seems. This is probably why it remained unnoticed for such a long time that by a combination of ingredients from both sides the CM equations become nearly trivial.

Even the expansion of $\langle e^{i\text{Tr}(XY)} \rangle$ in terms of Hi-Jack polynomials, see equation (40), can be found already in literature [22]. Whether this expansion is better adopted to random matrix problems as the existing expansion (15) in terms of Jack polynomials has still to be explored.

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Appendix A. The radial part of the momentum operator

For Hermitian matrices equation (12) reads, written in components,

$$x_{ij} = \sum_n r_{in} r_{jn}^* x_n. \quad (\text{A.1})$$

Introducing the notation

$$y_k = \begin{cases} x_k & k \leq N \\ \alpha_k & k > N \end{cases} \quad (\text{A.2})$$

where the α_k are the $N(N-1)$ angular variables, we have

$$\frac{\partial x_{ij}}{\partial y_k} = J_{k,ij} = \sum_{nm} r_{in} r_{jm}^* \hat{J}_{k,nm} \quad (\text{A.3})$$

where

$$\hat{J}_{k,nm} = \begin{cases} \delta_{kn} \delta_{km} & n = m \\ S_{k,nm} (x_m - x_n) & n \neq m \end{cases} \quad (\text{A.4})$$

and

$$S_{k,nm} = -S_{k,mn} = \sum_i r_{in}^* \frac{\partial r_{im}}{\partial \alpha_k}. \quad (\text{A.5})$$

It follows:

$$\begin{aligned} \nabla_{ij} &= \frac{\partial}{\partial x_{ji}} = \sum_k (J^{-1})_{ij,k} \frac{\partial}{\partial y_k} \\ &= \sum_{knm} r_{in} r_{jm}^* (\hat{J}^{-1})_{nm,k} \frac{\partial}{\partial y_k}. \end{aligned} \quad (\text{A.6})$$

This may be written as

$$\nabla_{ij} = \sum_{nm} r_{in} \tilde{\nabla}_{nm} r_{jm}^* \quad (\text{A.7})$$

where

$$\tilde{\nabla}_{nm} = \delta_{nm} \frac{\partial}{\partial x_n} + (1 - \delta_{nm}) \left[\sum_k (\hat{J}^{-1})_{nm,k} \frac{\partial}{\partial \alpha_k} - \frac{1}{x_n - x_m} \right]. \quad (\text{A.8})$$

Using representation (A.7) for the matrix elements of ∇ , the radial part of $\text{Tr}(X - \nabla)^k$ is now easily calculated. We obtain after a number of elementary steps

$$A_+^k = \sum_{nm} [(X_D - D)^k]_{nm} \quad (\text{A.9})$$

where

$$D_{nm} = \delta_{nm} \left(\frac{\partial}{\partial x_n} + \sum_k' \frac{1}{x_n - x_k} \right) - \frac{1 - \delta_{nm}}{x_n - x_m}. \quad (\text{A.10})$$

Equation (A.10) holds for Hermitian matrices, i.e. matrices taken from the unitary ensemble. Generalizing the calculation to the other ensembles we obtain equation (27).

Appendix B. Proof of relation (33)

The D_{nm} and Δ_{rad} (see equations (27) and (28)) obey the commutation rule

$$[\Delta_{\text{rad}}, D_{nm}] = \beta \sum_k' \left[(D_{nk} - D_{nm}) \frac{1}{(x_k - x_m)^2} - \frac{1}{(x_n - x_k)^2} (D_{km} - D_{nm}) \right] \quad (\text{B.1})$$

as can be verified directly from the definitions. The relation is true both for $n = m$ and $n \neq m$. Introducing the matrix S with elements

$$S_{nm} = \frac{1 - \delta_{nm}}{(x_n - x_m)^2} - \delta_{nm} \sum_k' \frac{1}{(x_n - x_k)^2} \quad (\text{B.2})$$

equation (B.1) may be written more concisely as

$$[\Delta_{\text{rad}}, D_{nm}] = \beta [D, S]_{nm}. \quad (\text{B.3})$$

Further commutation rules following directly from the definitions are

$$[\Delta_{\text{rad}}, x_n] = -2D_{nn} \quad (\text{B.4})$$

$$\left[\sum_k x_k^2, D_{nm} \right] = -2\delta_{nm} x_n. \quad (\text{B.5})$$

Introducing the operator A_+ with the components

$$(A_+)_{nm} = x_n \delta_{nm} - D_{nm} \quad (\text{B.6})$$

one obtains by combining equations (B.3)–(B.5)

$$[H, (A_+)_{nm}] = (A_+)_{nm} - \frac{\beta}{2} [A_+, S]_{nm}. \quad (\text{B.7})$$

Now the required commutator (33) can be calculated as

$$\begin{aligned} [H, A_+^k] &= \sum_{nm} [H, [(A_+)^k]_{nm}] \\ &= \sum_{v=0}^{k-1} \sum_{ijnm} [(A_+)^v]_{ni} [H, (A_+)_{ij}] [(A_+)^{(k-v-1)}]_{jm} \\ &= k A_+^k - \frac{\beta}{2} \sum_{v=0}^{k-1} \sum_{ijnm} [(A_+)^v]_{ni} [A_+, S]_{ij} [(A_+)^{(k-v-1)}]_{jm}. \end{aligned} \quad (\text{B.8})$$

The second term on the right-hand side does not contribute as is easily seen, and we are left with equation (33). \square

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